Abstract

Here, it is shown that the ‘traditional’ approach to variable bubble-point problems is not consistent, because it does not incorporate important kinds of shocks that occur in black-oil models. This paper contains an exhaustive analysis of shocks that black-oil models may develop and it is shown that a considerable variety of them may indeed occur; a bifurcation mechanism is also exhibited. Except for the class of shocks described by Buckley-Leverett theory, all other kinds may be developed even if capillary forces are present. Modelers must be aware of these pathologies in order to handle them properly. This paper also supplies a basis for the development of consistent black-oil models, applicable to variable bubble-point reservoirs. The analysis and results here presented, also contribute to a better understanding of limitations imposed by the simplifying assumptions of black-oil models.

Introduction

When applying black-oil models to variable bubble-point problems, it is frequently assumed that the bubble-point may vary inside the undersaturated region [1-4]. However, such assumption is incorrect because it contradicts the basic postulates on which black-oil models are built. Indeed, such postulates do not include molecular diffusion, nor mechanical dispersion, and it has been shown that a consequence of such omission is the bubble-point conservation law, according to which, when a gas-phase is not present, oil-particles conserve their gas content (disolved gas:oil ratio). However, such restrictions are not respected in the ‘traditional’ approach to modeling variable bubble-point reservoirs [1-4], as it is explained in Section 4 of this article.

To overcome this inconsistency, when bubble-point is variable, it is necessary to introduce shocks in which the bubble-point—the solution gas:oil ratio: $R_g$—is discontinuous. However, this is not done; in the ‘traditional’ approach [1-4], since the classical Buckley-Leverett theory was developed [5-9], modelers are prepared to deal with discontinuities of the saturation, but jumps of $R_g$ are not included.

In addition, the bubble-point conservation law imposes very severe restrictions to the manners in which the dissolved gas:oil ratio of an oil-particle can vary, when a gas-phase is absent; physically, it means that when a gas-phase is not present, two oil-particles cannot exchange dissolved gas, even if they are very close. This property in turn, produces a propensity of black-oil models to develop shocks, which becomes apparent in problems with variable bubble-point. The classical Buckley-Leverett theory [5-9] describes an important class of shocks that occur in black-oil or beta-models. However, this is not by any means the only kind of shocks that black-oil models may develop. On the contrary—as will be shown in the present paper—a considerable variety of shocks may occur in such models. In addition, there are processes that produce bifurcations of the shocks.

To make our discussion more systematic, this paper aims to carry out an exhaustive analysis of the different kinds of shocks that can occur in black-oil models, and the conditions under which they are generated; also, conditions under which they bifurcate. Such analysis is required to develop a consistent approach to variable bubble-point reservoirs and its results are clearly relevant in the numerical treatment of reservoirs, since the modeler must be aware of such pathologies when they occur, in order to handle them properly. In particular, previous numerical treatments of variable bubble-point systems did not include these features [1-4]. When dealing with variable bubble-point problems, in general, in which free gas may, or may not, coexist with liquid oil, the region of definition of the problem is made of three parts (Fig. 1):
The importance of our results is clear: they indicate a lack of consistency of the 'traditional' approach to variable bubble-point problems [1-4], because it does not incorporate discontinuities in the bubble-point (or solution gas:oil ratio). Thus, there is an urgent need to overcome this situation, since numerical results obtained using inconsistent models are unwarranted. However, our results supply a guide for this purpose and, in addition, another contribution is being prepared [13], to examine these matters more thoroughly -both numerical and theoretical implications.

The present article is organized as follows: In Section 2, the black-oil model hypotheses are made explicit and some notation is introduced; shock velocities are discussed in Section 3; shock generation and the bifurcation mechanism, are presented in Section 4; and the conclusions and a summary of results are given in Section 5. Mathematical derivations required for the developments are given in the Appendix. To make the results more general and increase their interest, the analysis is carried out in 3-dimensional space throughout the paper, except for an illustrative example presented in Section 4.

THE BLACK-OIL MODEL

For simplicity, we consider a "black-oil" or "beta" model, consisting of two phases, liquid oil and gas -whose Darcy velocities are denoted by \( u_o \) and \( u_g \), respectively-, based on the usual assumptions: 

a).- Gas is soluble in liquid oil; i.e., the gas phase consists of only one component, while the liquid oil is made up of two components (dissolved gas and non-volatile oil);

b).- No physical diffusion is present. This excludes both molecular diffusion and that induced by the randomness of the porous medium (mechanical dispersion).

In addition, the multiphase form of Darcy's Law is adopted; in particular, pressures are assumed to be continuous everywhere. Black-oil models, in general, include the possibility of non-vanishing capillary pressure, as is done here.

We use the notations \( \rho_o \) and \( \rho_{dg} \), for the effective densities of non-volatile oil and dissolved gas, respectively, together with the relation:

\[
\rho_{dg} = R_s \rho_o, \quad \text{where} \quad R_s = \frac{\rho_{STC}}{\rho_{STC}^s} \tag{1}
\]

The factor \( R_s \) is the "solution gas:oil ratio" [1]. In the following discussions, in addition to Darcy velocities "particle velocities" for each phase will be considered in some instances; they are defined by

\[
u_o = u_o \rho_o, \quad \nu_g = u_g \rho_{dg}
\]
\[ \nu_a = u_a / \phi S_a; \quad \alpha = 0 \text{ and } g \]  \hspace{1cm} (2)

Attention will be restricted to situations in which a gas front divides the region of study into two subregions (Fig. 1): one in which free gas is present (the "gas region") and the other one in which there is only undersaturated oil (the "undersaturated oil region"). To be specific, at the gas front, the unit normal vector \( \hat{n} \) will be taken as pointing towards the gas region. When considering discontinuities across a surface \( \Sigma \), the velocity of \( \Sigma \) will be \( \nu_\Sigma \), and the unit normal vector will be taken pointing towards the positive side. In particular, at the gas front the gas-side is the positive side (Fig. 1). Also, square brackets stand for the "jump" of a function; thus, for example: \( [R_s] = R_s^+ - R_s^- \).

**SHOCK VELOCITIES**

To be systematic and exhaustive in the analysis of shocks, it is necessary to consider three cases.

Case A.- The shock occurs at a gas front, so that the gas-phase is present at one side of the shock only. In this case \( R_s \) and the saturation may be discontinuous. Using Eq. (A7), of the Appendix, it is clear that

\[ (\nu_\Sigma - \nu_o^+) \cdot \hat{n} = \varepsilon (\nu_g - \nu_o^+) \cdot \hat{n}, \]  \hspace{1cm} (3a)

where

\[ \varepsilon = \frac{1}{1 + [R_s] \frac{\rho_o S_o}{\rho_g S_g}} \]  \hspace{1cm} (3b)

For case A, two possible situations must be distinguished.

A gas front advancing into a region of undersaturated oil.

This situation is characterized by \( (\nu_o^+ - \nu_g^-) \cdot \hat{n} > 0 \) and \( [R_s] = R_s^+ - R_s^- > 0 \), i.e., \( [R_s] = 1 \), because the oil is saturated in the gas-side. Thus, the parameter \( \varepsilon \) is a retardation factor, since it satisfies the condition \( \varepsilon > 1 \).

A gas front receding from a region of undersaturated oil.

This situation is characterized by the fact that \( (\nu_o^+ - \nu_g^-) \cdot \hat{n} < 0 \) and \( [R_s] = 0 \), necessarily, because as the gas front recedes it leaves saturated oil behind. Thus the oil is saturated on both sides of the gas front and \( R_s \) is continuous across the front, since so is the pressure. The only discontinuous variable is \( S_o \). Eq. (3b) implies that \( \varepsilon = 1 \), so that there is no retardation and the gas front recedes with the velocity of the gas particles.

Case B.- The shock occurs in the unsaturated oil region, so that the gas-phase is absent from both sides of the shock.

The only possible discontinuous variable is \( R_s \), since \( S_o = 1 \), at both sides of the shock. In the Appendix it is shown that in this case:

\[ \nu_\Sigma \cdot \hat{n} = \nu_o \cdot \hat{n}, \quad \text{on } \Sigma. \]  \hspace{1cm} (4)

Situations in which shocks of these characteristics may occur are discussed in the next section.

Case C.- The shock occurs in the gas region, so that the gas-phase is present at both sides of the shock. Oil is saturated at both sides of the shock; \( R_s \) is continuous by virtue of the pressure continuity. The only possible discontinuous variable is the saturation. According to the Appendix, in this case:

\[ \nu_\Sigma \cdot \hat{n} = \frac{[u_g] \cdot \hat{n}}{S_o / \phi}. \]  \hspace{1cm} (5)

A special case of this equation is the immiscible and incompressible case considered by the classical Buckley-Leverett theory, for which Eq. (5) becomes the well-known relation (see for example [6] and [11]):

\[ \nu_\Sigma \cdot \hat{n} = \frac{f_g [u_g]}{S_o} \]  \hspace{1cm} (6a)

where

\[ u_g = u_o + u_g \]  \hspace{1cm} and \[ u_g = f_g u_R \]  \hspace{1cm} (6b)

**SHOCK GENERATION**

In this section the mechanisms of shock generation and bifurcation are discussed.

**IN AN UNSATURATED REGION**

"Oil particles" enjoy the following property in this region:

**BUBBLE POINT CONSERVATION LAW**

In the absence of a gas phase, oil particles conserve their bubble-point.
Proof. Equation (A4), of the Appendix, applies. Thus the "material particle derivative" of $R_s$ vanishes. Clearly this implies that $R_s$ (i.e., the bubble-point), remains constant on liquid oil particles.

As mentioned in Section 3, the only variable, possibly discontinuous, is $R_s$. In contrast to the "gas region", where oil is necessarily saturated and $R_s$ is uniquely determined by pressure, when the gas phase is absent the liquid oil will usually be undersaturated and $R_s$ can take any value below the saturation curve (Fig. 2).

**FIGURE 2.- Paths in the $R_s - p_o$ plane.**

This is related with the initial conditions associated with well-posed problems. When free gas is present, the oil is saturated and the pressure determines $R_s$ and it is not necessary to include this parameter in the initial conditions. On the contrary, if the oil is undersaturated $R_s$ is not determined by pressure and must be prescribed as an initial condition. In particular, if the prescribed initial or boundary values of $R_s$ are discontinuous, a shock in the unsaturated region would be introduced because, by virtue of the "bubble-point conservation law", the liquid oil particles will retain their $R_s$ values; such discontinuity will propagate with velocity $v_s$, as required by Eq. (4). Therefore, shocks of this kind can be produced by the initial conditions and possibly by the boundary conditions.

To be exhaustive, in the enumeration of the general mechanisms of this kind of shocks, it must be mentioned that... To be exhaustive, in the enumeration of the general mechanisms of this kind of shocks, it must be mentioned that... Such process constitutes the "bifurcation mechanism" that... As mentioned in Section 3, the only variable, possibly discontinuous, is $R_s$. In contrast to the "gas region", where oil is necessarily saturated and $R_s$ is uniquely determined by pressure, when the gas phase is absent the liquid oil will usually be undersaturated and $R_s$ can take any value below the saturation curve (Fig. 2).

**AT A GAS-FRONT**

The bubble-point conservation law is a very restrictive condition and, at a gas-front, shocks are generated when undersaturated particles reach the front and become suddenly saturated. This would happen at an advancing gas-front, not at a gas-front that is receding from a region occupied by undersaturated oil, as is explained next.

Due to "bubble-point conservation law", on an oil particle, paths of $R_s$ in the $R_s - p_o$ plane consist of fragments of the saturation curve or of horizontal segments, only (Fig. 2). The first ones take place in periods spent by the particle in regions where the gas phase is present, while the latter ones correspond to periods spent by the particle in undersaturated regions where the gas-phase is necessarily absent. Thus, if a particle starts at state "n" (Fig. 2a), so that it is undersaturated initially, and if it is then depressurized, it moves along a horizontal line towards the left until it reaches the saturation curve. If depressurization of the particle continues, it bubbles and liberates gas. If depressurization is stopped, the free gas is removed and the oil is repressurized to the state of the particle in the $R_s - p_o$ plane along a horizontal line, this time towards the right; it finally reaches a state such as "n+1" (Fig. 2a). This path is reversible: we could start at state "n+1" and by successive depressurization and represurization, reach state "n". The point at which the mixture leaves the saturation curve when it is repressurized depends on the amount of free gas available. In actual reservoirs, this amount of gas is supplied by the gas phase, which in turn is determined by the relative motion of the liquid oil phase with respect to it.

On the other hand, on the $R_s - p_o$ plane the states of an oil particle cannot follow a path such as the one joining states "n" and "n+1" (Fig. 2b), since this would imply that $R_s$ changes without reaching the bubble point. That is, $R_s$ would change when the gas phase is absent and the bubble-point conservation law would be violated. However, in the...
'traditional' approach to variable bubble-point reservoirs, such restriction is not respected [1-4].

At an advancing gas-front
At first glance, the previous discussion suggests that in a beta model the only way in which an undersaturated oil particle may become saturated is by depressurization to the bubble point. This would imply that the beta model is a very limited model, especially when considering problems in which the bubble-point varies, since it cannot mimic the processes by which an undersaturated particle of oil receives gas from other particles. However, such limitation is somewhat relaxed by the fact that in a beta model an oil particle may become saturated, in another manner: it may follow a discontinuous path on the $R_s-p_o$ plane, such as SH-SH', in Fig. 2c. This corresponds to an oil particle which is initially undersaturated (point "n") so that the gas phase is absent. At some point the oil particle is reached by a gas-front (point SH) and becomes suddenly saturated (point SH'); under further pressurization $R_s$ moves along the saturation curve. Such a path has the discontinuity SH-SH' and therefore $[R_s] \neq 0$, there. In actual reservoir models, this clearly corresponds to a discontinuous front or shock. In this case, the shock itself constitutes a mechanism for transferring gas from the gas-phase to the oil-particles.

At an advancing gas-front, due to the bubble-point conservation principle, a shock of this kind generally will occur even if the initial conditions are continuous. This is because the continuity of the initial values does not prevent oil particles, carrying values of $R_s$ below the saturation value, from reaching the gas-front, where $R_s$ necessarily equals the saturation value. This is the mechanism of shock generation at an advancing gas-front. Observe, that in general at such a shock both $[R_s] \neq 0$ and $[S_o] \neq 0$.

At a receding gas front
At a front that is receding, on the contrary, $R_s$ is necessarily continuous because the gas phase leaves saturated oil behind it, as it goes away. Since $R_s$ is continuous, the only discontinuous variable is the saturation. Setting $[R_s] = 0$, in Eq. (3b), yields $\varepsilon = 1$, which implies $v_{\varepsilon} \cdot n = v^g \cdot n$; i.e., the gas-front moves together with the gas particles that constitute it.

A BIFURCATION MECHANISM
Note that if an advancing gas-front changes its sense of motion, thus becoming a receding one, at the point where it stops and starts to recede, a discontinuity of $R_s$ at the oil phase remains. Such shock will move with velocity $v_o$, as pointed out in Section 3; thus, it will move towards the interior of the "undersaturated region" and, at later times, it will be located inside that region. This is the additional generation mechanism that was mentioned in the discussion of those shocks.

On the other hand, this analysis also indicates that where an advancing front -carrying discontinuities in both $R_s$ and $S_o$- stops and starts to recede, the shock "bifurcates", giving rise to two shocks: one in which the only discontinuous variable is the saturation and the other one in which the only discontinuous variable is the bubble-point. This phenomenon is illustrated in Fig. 3: if the front, when it starts to recede, is

![Figure 3](image-url)

**FIGURE 3.** The bifurcation mechanism:

a) The bifurcation point at $x_B$.

b) The two shocks, after bifurcation.

located at $x_B$, then the discontinuities of $R_s$ and of $S_o$ are together there (Fig. 3a). However, as the receding motion of the front progresses, these discontinuities separate because they move with different velocities; this is shown in Fig. 3b, where the discontinuity of the bubble-point ($R_s$) is located at $x_{EB}$, to the left of $x_B$, while the discontinuity of the saturation is located at $x_{ES}$, to the right of $x_B$. Fig. 4 next- shows the illustrative example to be explained, corresponding to the illustrative example to be explained in space-time.

An illustrative example
The bifurcation point and the paths of the discontinuities, in the x-t plane, corresponding to a simple one-dimensional-numerical example, are illustrated in Fig. 4.

FIGURE 4.-Shocks and bifurcation for the illustrative example.

The shocks are: \( \Sigma SR, \Sigma S \) and \( \Sigma R_n \), where the discontinuous variables are: \( S \) and \( R \), \( S \) and \( R \), respectively.

The parameter values used in this example are given in Table 1; also a simplified equation of state was assumed (see, Fig. 5), and \( p \) satisfies \( p \) = \( p^* \). A positive pressure gradient \( \partial p / \partial x \)

\[ \frac{S^*_o}{S^*_g} \] as well as, \[ \left[ R_o \right] = \left[ R_g \right] \frac{\rho_{oSTC}}{\rho_{gSTC}} \], which for this case take the values 2,350, 4 and .0178, respectively, giving a pretty large product: 167.32. Due to the drastic retardation effect, the velocity of the receding gas-front is much larger than that of the advancing one and the scales involved when the fronts are advancing and receding, are quite different, as can be clearly appreciated in Fig. 4. On the other hand, for this example, the contrast between \( v_{LR} \) and \( v_{SR} \), is not as sharp: \( v_{LR} / v_{SR} \equiv 0.2 \). Of course, if these parameters varied, the behavior of the fronts and the bifurcation may changed very much.

### TABLE 1: Parameter Values for the Illustrative Exam PROPERTIES

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_o )</td>
<td>47 pound/ft(^3)</td>
</tr>
<tr>
<td>( \rho_g )</td>
<td>.02 pound/ft(^3)</td>
</tr>
<tr>
<td>( \mu_o )</td>
<td>.3 cp</td>
</tr>
<tr>
<td>( \mu_g )</td>
<td>.03 cp</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \kappa )</td>
<td>100 m darcies</td>
</tr>
<tr>
<td>( \kappa_{so} )</td>
<td>0.5</td>
</tr>
<tr>
<td>( \kappa_{sg} )</td>
<td>0.4</td>
</tr>
</tbody>
</table>

### DATA

- \( S_o = 0.2 \)
- \( S^*_o = 0.8 \)
- \( R_o \) = \( 78 \times 10^{-2} \)

\( v_s = 5.34(t-t_B) \)

\( x_o = 800; x_g = 600.53; t_B = 112.4 \)

### DERIVED PARAMETERS

\( v_o^* = 16.7 \times 10^{-2}(t-t_B) \)

\( \epsilon = 5.9 \times 10^{-3} \)

ADVANCING \( (t(t_B) \)

\( v_o^- = 73 \times 10^{-1}(t-t_B) \)

\( v_{LSR} = 0.0316(t-t_B) \)

\( x_{LSR} = 600.53 + 1.58 \times 10^{-2}(t-t_B) \)

RECEDING \( (t(t_B) \)

\( v_o^- = 1.2(t-t_B) \)

\( v_{LS} = 5.34(t-t_B); v_{SR} = 1.2(t-t_B) \)

\( x_{LS} = 600.53 + 2.67 (t-t_B)^2 \)

\( x_{SR} = 600.53 + .6 (t-t_B)^2 \)

Note: Distances are in feet and times in days.

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As it would be expected, big inaccuracies are obtained in the evaluation of gas production rates and other quantities of interest in the oil industry, when shocks and bifurcations are not incorporated in the models. This has been confirmed through numerical examples in research that is underway [13]. Thus, the relevance of the phenomena addressed by this paper, is clear.

**AT REGIONS WHERE FREE GAS IS PRESENT**

This kind of shocks are generated by a mechanism that was originally described by Buckley and Leverett [5-9] and further discussed by many authors. They occur when characteristic curves carrying different values of saturation intersect, giving rise to multi-valued solutions which are not physically admissible. A very clear discussion of this process was presented by Sheldon and Cardwell [8,9]. A recent account, from a present-day perspective, is given in [11].

Note that, as it is generally recognized, in a region where free-gas is present, such shocks only develop when capillary pressure is neglected. If capillary pressure is present, the continuity condition for the pressure has to be satisfied by both phases. This is possible only if the capillary pressure is continuous. In turn, this implies the continuity of saturation, since capillary pressure is a continuous function of $S_o$. However, other kinds of discontinuities that were discussed above may be generated even if capillary pressure is incorporated in the model. This is because in the other cases considered, the continuity condition for the gas pressure is not enforced at the shock, since the gas pressure is not defined at least at one side of it.

**CONCLUSIONS**

In the developments presented in this paper, it has been shown that the omission of diffusion mechanisms -both, molecular diffusion and mechanical dispersion- in black-oil models leads to the "bubble-point conservation principle", according to which: in the absence of a gas-phase, oil-particles conserve their bubble-point. Due to this result, paths in the $R_s - p_o$ plane, usually admitted in the 'traditional' treatment of variable bubble-point reservoirs, are actually forbidden and in order to develop a consistent approach, it is necessary to introduce shocks in which not only the saturation may be discontinuous, but also the bubble-point.

An exhaustive analysis of the shocks that black-oil models may develop, yielded a considerable variety of them, and also a bifurcation mechanism. Although one of such shocks is described by Buckley-Leverett theory, the others are not, and they may occur even if capillary forces are taken into account. This propensity of black-oil models to develop shocks, is related to the omission of diffusion mechanisms.

In summary, shocks are classified according to whether they occur at: 1) a gas region, where gas is present at both sides of the shock, 2) a gas-front, and 3) a region occupied by undersaturated oil. Simple expressions are given for the velocities of each kind. For shocks of type 1), only the saturation can jump and they are essentially described by Buckley-Leverett theory. In case 3), only the bubble-pair (i.e., the solution oil:gas ratio ($R_s$)) can jump. In case 2), an advancing gas-front, both $S_o$ and $R_s$ are discontinuous. A receding gas-front only the saturation can jump. In addition when an advancing gas-front changes its sense of motion an starts to recede, the shock bifurcates, giving rise to two shocks, one moving with the oil velocity, where only $R_i$ is discontinuous, and the other one with the velocity of the gas where only $S_o$ jumps.

The importance of the results presented is clear: they indicate a lack of consistency in the 'traditional' approach to variable bubble-point problems [1-4], because it does not incorporate discontinuities in the bubble-point (or solution gas-ratio) This point is critical, since numerical results obtained using inconsistent models are unwarranted. Indeed, it is natural to expect big inaccuracies in the evaluation of gas production rates and other quantities of interest in the oil industry, when shocks and bifurcations are not incorporated in the model that are applied to variable bubble-point problems, and this has been confirmed through numerical examples in research that is underway [13]. However, our results supply a guide for constructing consistent black-oil models, applicable to variable bubble-point problems, and another contribution is being prepared [13], to examine these matters more thoroughly -both numerical and theoretical implications.

**REFERENCES**


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The results that follow, were originally derived from first principles - using the general "jump" conditions of continuous systems (see, for example, Chapter 1 of [14]), as was done in [12]. However, the manner of presenting it here - more suitable for a Petroleum Engineering audience - owes much to the referee of this paper. This is thankfully acknowledged at this point.

Firstly, Eq. (2) is recalled, since it will be frequently used in what follows. As it is customary in Petroleum Engineering, the governing equations of the black-oil model are written as:

\[
\begin{align*}
\phi S_o \rho_o \left( \frac{\overline{R_s}}{\rho_o} \right) + \text{div} \left( \rho_o \overline{u_o} \right) &= 0 \\
\phi S_o \rho_o \left( \frac{\overline{R_s}}{\rho_o} \right) + \text{div} \left( \rho_o \overline{R_s} \overline{u_o} \right) + \phi S_g \rho_g \left( \frac{\rho_g}{\phi S_g} \right) + \text{div} \left( \rho_g \overline{u_g} \right) &= 0
\end{align*}
\]  
(A1a)

(A1b)

Mass conservation implies these equations, as well as the shock conditions. These latter conditions, can be thought as limit forms of (A1); they are [12]:

\[
\begin{align*}
\rho_o^+ (u_o - \phi \overline{v_S} S_o)^+ \cdot \mathbf{n} &= \rho_o^- (u_o - \phi \overline{v_S} S_o)^- \cdot \mathbf{n} \\
\rho_g^+ (\rho_g \overline{v_g})^+ \cdot \mathbf{n} &= \rho_g^- (\rho_g \overline{v_g})^- \cdot \mathbf{n}
\end{align*}
\]  
(A2a)

Combining Eqs. (A1), it is seen that

\[
\begin{align*}
\phi S_o \rho_o \left( \frac{\overline{R_s}}{\rho_o} \right) + \overline{v_o} \cdot \text{grad} \left( \overline{R_s} \right) = 0 \\
+ \phi S_g \rho_g \left( \frac{\rho_g}{\phi S_g} \right) + \text{div} \left( \rho_g \overline{u_g} \right) &= 0
\end{align*}
\]  
(A3)

When the gas phase is absent, this equation reduces to

\[
\begin{align*}
\phi S_o \rho_o \left( \frac{\overline{R_s}}{\rho_o} \right) + \overline{v_o} \cdot \text{grad} \left( \overline{R_s} \right) = 0
\end{align*}
\]  
(A4)

In words: "the material oil-particle derivative vanishes".

Case A.- This is characterized by \( S_g^- = 0 \), and the right side of (A2b) vanishes. Using Eq. (2), it is obtained

\[
\begin{align*}
\left( \frac{\rho_o}{\phi S_o} \right) \overline{R_s} + \text{div} \left( \rho_g \overline{v_g} \right) = 0
\end{align*}
\]  
(A5)

Making use of the relation

\[
\begin{align*}
\overline{v_g} - \overline{v_L} &= \overline{v_o} + \overline{v_o} - \overline{v_L}
\end{align*}
\]  
(A6)

yields

\[
\begin{align*}
1 + \left( \frac{\rho_o}{\phi S_o} \right) \overline{R_s} \overline{v_o} - \overline{v_L} \cdot \mathbf{n} = (\overline{v_o} - \overline{v_o}) \cdot \mathbf{n}
\end{align*}
\]

Thus, Eqs. (3) follow.

Case B.- In this case the gas phase is absent, so that the equations involving it, in Eq. (A2b), vanish. Hence, this equation clearly implies (4), since \([ R_s ] \neq 0\).

Case C.- The oil is saturated and \([ R_s ] = 0\), since the pressure is continuous. Thus, Eq. (A2b) reduces to (5).

BIOGRAPHICAL PROFILE

Ismael Herrera is a mathematician - with a strong background in pure mathematics- devoted to applications of mathematics: civil and petroleum engineering, geophysics and water resources, mainly. He got his Ph.D. in Applied Mathematics from Brown University, after having carried out undergraduate studies in Chemistry, Physics and Mathematics, at UNAM (Mexico).